

K-HARMONIC CURVES INTO A RIEMANNIAN MANIFOLD WITH CONSTANT SECTIONAL CURVATURE

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ABSTRACT. In [5], J.Eells and L. Lemaire introduced k -harmonic maps, and T. Ichiyama, J. Inoguchi and H. Urakawa [1] showed the first variation formula. In this paper, we describe the ordinary differential equations of 3-harmonic curves into a Riemannian manifold with constant sectional curvature, and show biharmonic curve is k -harmonic curve ($k \geq 2$).

Introduction

Theory of harmonic maps has been applied into various fields in differential geometry. The harmonic maps between two Riemannian manifolds are critical maps of the energy functional $E(\phi) = \frac{1}{2} \int_M \|d\phi\|^2 v_g$, for smooth maps $\phi : M \rightarrow N$.

On the other hand, in 1981, J. Eells and L. Lemaire [5] proposed the problem to consider the k -harmonic maps: they are critical maps of the functional

$$E_k(\phi) = \int_M e_k(\phi) v_g, \quad (k = 1, 2, \dots),$$

where $e_k(\phi) = \frac{1}{2} \|(d + d^*)^k \phi\|^2$ for smooth maps $\phi : M \rightarrow N$. G.Y. Jiang [4] studied the first and second variation formulas of the bi-energy E_2 , and critical maps of E_2 are called *biharmonic maps*. There have been extensive studies on biharmonic maps.

Recently, in 2009, T. Ichiyama, J. Inoguchi and H. Urakawa [1] studied the first variation formula of the k -energy E_k , whose critical maps are called k -harmonic maps. Harmonic maps are always k -harmonic maps by definition. In this paper, we study k -harmonic curves.

In §1, we introduce notation and fundamental formulas of the tension field.

In §2, we show biharmonic curves into a Riemannian manifold with constant sectional curvature is always k -harmonic curves.

1. PRELIMINARIES

Let (M, g) be an m dimensional Riemannian manifold, (N, h) an n dimensional one, and $\phi : M \rightarrow N$, a smooth map. We use the following notation. The second fundamental form $B(\phi)$ of ϕ is a covariant differentiation $\tilde{\nabla} d\phi$ of 1-form $d\phi$, which is a section of $\odot^2 T^*M \otimes \phi^{-1}TN$. For every $X, Y \in \Gamma(TM)$, let

$$(1) \quad \begin{aligned} B(X, Y) &= (\tilde{\nabla} d\phi)(X, Y) = (\tilde{\nabla}_X d\phi)(Y) \\ &= \bar{\nabla}_X d\phi(Y) - d\phi(\nabla_X Y) = \nabla_{d\phi(X)}^N d\phi(Y) - d\phi(\nabla_X Y). \end{aligned}$$

Here, $\nabla, \nabla^N, \bar{\nabla}, \tilde{\nabla}$ are the induced connections on the bundles $TM, TN, \phi^{-1}TN$ and $T^*M \otimes \phi^{-1}TN$, respectively.

If M is compact, we consider critical maps of the energy functional

$$(2) \quad E(\phi) = \int_M e(\phi) v_g,$$

where $e(\phi) = \frac{1}{2}\|d\phi\|^2 = \sum_{i=1}^m \frac{1}{2}\langle d\phi(e_i), d\phi(e_i) \rangle$ which is called the *energy density* of ϕ , and the inner product $\langle \cdot, \cdot \rangle$ is a Riemannian metric h . The *tension field* $\tau(\phi)$ of ϕ is defined by

$$(3) \quad \tau(\phi) = \sum_{i=1}^m (\tilde{\nabla} d\phi)(e_i, e_i) = \sum_{i=1}^m (\tilde{\nabla}_{e_i} d\phi)(e_i).$$

Then, ϕ is a *harmonic map* if $\tau(\phi) = 0$.

And we define

$$(4) \quad \overline{\Delta} = \overline{\nabla}^* \overline{\nabla} = - \sum_{k=1}^m (\overline{\nabla}_{e_k} \overline{\nabla}_{e_k} - \overline{\nabla}_{\nabla_{e_k} e_k}),$$

is the *rough Laplacian*.

And we define \mathcal{R} as follows :

$$(5) \quad \mathcal{R}(V) := \sum_{i=1}^m R^N(V, d\phi(e_i))d\phi(e_i), \quad V \in \Gamma(\phi^{-1}TN),$$

where,

$$R^N(U, V) = \nabla_U^N \nabla_V^N - \nabla_V^N \nabla_U^N - \nabla_{[U, V]}^N, \quad U, V \in \Gamma(TN),$$

is the curvature tensor of (N, h) .

2. k -HARMONIC CURVES INTO A RIEMANNIAN MANIFOLD WITH CONSTANT SECTIONAL CURVATURE

In this section, we consider curves into a Riemannian manifold with constant sectional curvature. Then, we show the necessary and sufficient condition of 3-harmonic curve, and biharmonic curve is k -harmonic curve.

J. Eells and L. Lemaire [5] proposed the notation of k -harmonic maps. The Euler-Lagrange equations for the k -harmonic maps was shown by T. Ichiyama, J. Inoguchi and H. Urakawa [1]. We first recall it briefly.

Theorem 2.1 ([1]). *Let $k = 2, 3, \dots$. Then, we have*

$$\left. \frac{d}{dt} \right|_{t=0} E_k(\phi_t) = - \int_M \langle \tau_k(\phi), V \rangle v_g,$$

where

$$\tau_k(\phi) := J \left(\overline{\Delta}^{(k-2)} \tau(\phi) \right) = \overline{\Delta} \left(\overline{\Delta}^{(k-2)} \tau(\phi) \right) - \mathcal{R} \left(\overline{\Delta}^{(k-2)} \tau(\phi) \right),$$

and

$$\overline{\Delta}^{(k-2)} \tau(\phi) = \underbrace{\overline{\Delta} \cdots \overline{\Delta}}_{k-2} \tau(\phi).$$

As a corollary of this theorem, we have

Corollary 2.2 ([1]). *$\phi : (M, g) \rightarrow (N, h)$ is a k -harmonic map if*

$$(6) \quad \tau_k(\phi) := J \left(\overline{\Delta}^{(k-2)} \tau(\phi) \right) = \overline{\Delta} \left(\overline{\Delta}^{(k-2)} \tau(\phi) \right) - \mathcal{R} \left(\overline{\Delta}^{(k-2)} \tau(\phi) \right) = 0.$$

We say for a k -harmonic map to be *proper* if it is not harmonic.

Let us recall the definition of the Frenet frame.

Definition 2.3. The Frenet frame $\{e_i\}_{i=1,\dots,n}$ associated to a curve $\gamma : I \subset \mathbb{R} \rightarrow (N^n, \langle \cdot, \cdot \rangle)$, parametrized by arc length, is the orthonormalisation of the $(n+1)$ -uple $\{\nabla_{d\gamma(\frac{\partial}{\partial t})}^{N(k)} d\gamma(\frac{\partial}{\partial t})\}_{k=1,\dots,n}$, described by

$$\begin{aligned} e_1 &= d\gamma\left(\frac{\partial}{\partial t}\right), \\ \nabla_{d\gamma(\frac{\partial}{\partial t})}^N e_1 &= \kappa_1 e_2, \\ \nabla_{d\gamma(\frac{\partial}{\partial t})}^N e_i &= -\kappa_{i-1} e_{i-1} + \kappa_i e_{i+1} \quad (i = 2, \dots, n-1), \\ \nabla_{d\gamma(\frac{\partial}{\partial t})}^N e_n &= -\kappa_{n-1} e_{n-1}, \end{aligned}$$

where the functions $\kappa_1, \kappa_2, \dots, \kappa_{n-1}$ are called the curvatures of γ . Note that $e_1 = \gamma'$ is the unit tangent vector field along the curve.

First, we show the necessary and sufficient condition of k -harmonic curves into a Riemannian manifold with constant sectional curvature.

Proposition 2.4. *Let $\gamma : I \rightarrow (N^n, \langle \cdot, \cdot \rangle)$ be a smooth curve parametrized by arc length from an open interval of \mathbb{R} into a Riemannian manifold $(N^n, \langle \cdot, \cdot \rangle)$ with constant sectional curvature K . Then, γ is k -harmonic if and only if,*

$$(7) \quad (\nabla_{\gamma'}^N \nabla_{\gamma'}^N)^{k-1} \tau(\gamma) - K \{ (\nabla_{\gamma'}^N \nabla_{\gamma'}^N)^{k-2} \tau(\gamma) - \langle \gamma', (\nabla_{\gamma'}^N \nabla_{\gamma'}^N)^{k-2} \tau(\gamma) \rangle \gamma' \} = 0.$$

Proof.

$$\begin{aligned} \overline{\Delta} \tau(\gamma) &= (-1)^{k-1} (\nabla_{\gamma'}^N \nabla_{\gamma'}^N)^{k-1} \tau(\gamma), \\ \mathcal{R}(\overline{\Delta}^{k-2} \tau(\gamma)) &= K \{ (-1)^{k-2} (\nabla_{\gamma'}^N \nabla_{\gamma'}^N)^{k-2} \tau(\gamma) - \langle \gamma', (-1)^{k-2} (\nabla_{\gamma'}^N \nabla_{\gamma'}^N)^{k-2} \tau(\gamma) \rangle \gamma' \}. \end{aligned}$$

Therefore, we have Proposition 2.4. □

Using Proposition 2.4, we show the necessary and sufficient condition of biharmonic curve and 3-harmonic curve, respectively.

Proposition 2.5. *Let $\gamma : I \rightarrow (N^n, \langle \cdot, \cdot \rangle)$ be a smooth curve parametrized by arc length from an open interval of \mathbb{R} into a Riemannian manifold $(N^n, \langle \cdot, \cdot \rangle)$ with constant sectional curvature K . Then, γ is proper biharmonic if and only if,*

$$(8) \quad \begin{cases} \kappa_1^2 + \kappa_2^2 = K, \\ \kappa_1 = \text{constant} \neq 0, \\ \kappa_2 = \text{constant}, \\ \kappa_2 \kappa_3 = 0. \end{cases}$$

Proof. $\tau(\gamma) = \kappa_1 e_2$. So we calculate $(\nabla_{\gamma'}^N \nabla_{\gamma'}^N)(\kappa_1 e_2)$ as follows.

$$\begin{aligned} (9) \quad (\nabla_{\gamma'}^N \nabla_{\gamma'}^N)(\kappa_1 e_2) &= -3\kappa_1 \kappa_1' e_1 + (\kappa_1'' - \kappa_1^3 - \kappa_1 \kappa_2^2) e_2 + (2\kappa_1' \kappa_2 + \kappa_1 \kappa_2') e_3 + \kappa_1 \kappa_2 \kappa_3 e_4. \end{aligned}$$

Using Proposition 2.4, and $\kappa_1 \neq 0$, we have Proposition 2.5. □

Proposition 2.6. *Let $\gamma : I \rightarrow (N^n, \langle \cdot, \cdot \rangle)$ be a smooth curve parametrized by arc length from an open interval of \mathbb{R} into a Riemannian manifold $(N^n, \langle \cdot, \cdot \rangle)$ with constant sectional curvature K . Then, γ is 3-harmonic if and only if,*

$$(10) \quad \begin{cases} -2\kappa_1'\kappa_1'' - \kappa_1\kappa_1^{(3)} + 2\kappa_1^3\kappa_1' + \kappa_1\kappa_1'\kappa_2^2 + \kappa_1^2\kappa_2\kappa_2' = 0 \\ -15\kappa_1(\kappa_1')^2 - 10\kappa_1^2\kappa_1'' + \kappa_1^5 + 2\kappa_1^3\kappa_2^2 + \kappa_1^{(4)} - 6\kappa_1''\kappa_2^2 - 12\kappa_1'\kappa_2\kappa_2' \\ -3\kappa_1(\kappa_2')^2 - 4\kappa_1\kappa_2\kappa_2'' + \kappa_1\kappa_2^4 + \kappa_1\kappa_2^2\kappa_3^2 + K\{\kappa_1'' - \kappa_1^3 - \kappa_1\kappa_2^2\} = 0 \\ 4\kappa_1^{(3)}\kappa_2 - 9\kappa_1^2\kappa_1'\kappa_2 - 4\kappa_1'\kappa_2^3 - 6\kappa_1\kappa_2^2\kappa_2' + 6\kappa_1''\kappa_2' - \kappa_1^3\kappa_2' \\ + 4\kappa_1'\kappa_2'' + \kappa_1\kappa_2^{(3)} - 4\kappa_1'\kappa_2\kappa_3^2 - 3\kappa_1\kappa_2'\kappa_3^2 - 3\kappa_1\kappa_2\kappa_3\kappa_3' + K\{2\kappa_1'\kappa_2 + \kappa_1\kappa_2'\} = 0 \\ 6\kappa_1''\kappa_2\kappa_3 - \kappa_1^3\kappa_2\kappa_3 - \kappa_1\kappa_2^3\kappa_3 + 8\kappa_1'\kappa_2'\kappa_3 + 3\kappa_1\kappa_2''\kappa_3 - \kappa_1\kappa_2\kappa_3^3 \\ + 4\kappa_1'\kappa_2\kappa_3' + 3\kappa_1\kappa_2'\kappa_3' + \kappa_1\kappa_2\kappa_3'' - \kappa_1\kappa_2\kappa_3\kappa_4^2 + K\{\kappa_1\kappa_2\kappa_3\} = 0 \\ 4\kappa_1'\kappa_2\kappa_3\kappa_4 + 3\kappa_1\kappa_2'\kappa_3\kappa_4 + 2\kappa_1\kappa_2\kappa_3'\kappa_4 + \kappa_1\kappa_2\kappa_3\kappa_4' = 0 \\ \kappa_1\kappa_2\kappa_3\kappa_4\kappa_5 = 0 \end{cases}$$

Proof. We calculate $(\nabla_{\gamma'}^N \nabla_{\gamma'}^N)^2 \tau(\gamma)$ as follows.

$$\begin{aligned} & (\nabla_{\gamma'}^N \nabla_{\gamma'}^N)^2 \tau(\gamma) \\ &= (-10\kappa_1'\kappa_1'' - 5\kappa_1\kappa_1^{(3)} + 10\kappa_1^3\kappa_1' + 5\kappa_1\kappa_1'\kappa_2^2 + 5\kappa_1^2\kappa_2\kappa_2')e_1 \\ &+ (-15\kappa_1(\kappa_1')^2 - 10\kappa_1^2\kappa_1'' + \kappa_1^5 + 2\kappa_1^3\kappa_2^2 + \kappa_1^{(4)} \\ &- 6\kappa_1''\kappa_2^2 - 12\kappa_1'\kappa_2\kappa_2' - 3\kappa_1(\kappa_2')^2 - 4\kappa_1\kappa_2\kappa_2'' + \kappa_1\kappa_2^4 + \kappa_1\kappa_2^2\kappa_3^2)e_2 \\ &+ (4\kappa_1^{(3)}\kappa_2 - 9\kappa_1^2\kappa_1'\kappa_2 - 4\kappa_1'\kappa_2^3 - 6\kappa_1\kappa_2^2\kappa_2' \\ &+ 6\kappa_1''\kappa_2' - \kappa_1^3\kappa_2' + 4\kappa_1'\kappa_2'' + \kappa_1\kappa_2^{(3)} - 4\kappa_1'\kappa_2\kappa_3^2 \\ &- 3\kappa_1\kappa_2'\kappa_3^2 - 3\kappa_1\kappa_2\kappa_3\kappa_3')e_3 \\ &+ (6\kappa_1''\kappa_2\kappa_3 - \kappa_1^3\kappa_2\kappa_3 - \kappa_1\kappa_2^3\kappa_3 + 8\kappa_1'\kappa_2'\kappa_3 \\ &+ 3\kappa_1\kappa_2''\kappa_3 - \kappa_1\kappa_2\kappa_3^3 + 4\kappa_1'\kappa_2\kappa_3' \\ &+ 3\kappa_1\kappa_2'\kappa_3' + \kappa_1\kappa_2\kappa_3'' - \kappa_1\kappa_2\kappa_3\kappa_4^2)e_4 \\ &+ (4\kappa_1'\kappa_2\kappa_3\kappa_4 + 3\kappa_1\kappa_2'\kappa_3\kappa_4 + 2\kappa_1\kappa_2\kappa_3'\kappa_4 + \kappa_1\kappa_2\kappa_3\kappa_4')e_5 \\ &+ \kappa_1\kappa_2\kappa_3\kappa_4\kappa_5e_6 \end{aligned}$$

Using Proposition 2.4, and (9), we have Proposition 2.6. \square

We showed biharmonic curve is k -harmonic curve into 2-dimensional unit sphere [2]. We generalize this as following.

Theorem 2.7. *Let $\gamma : I \rightarrow (N^n, \langle \cdot, \cdot \rangle)$ be a smooth curve parametrized by arc length from an open interval of \mathbb{R} into a Riemannian manifold $(N^n, \langle \cdot, \cdot \rangle)$ with constant sectional curvature K . Then, biharmonic is k -harmonic ($k \geq 2$).*

Proof. By Proposition 2.5, γ is proper biharmonic if and only if

$$\begin{cases} \kappa_1^2 + \kappa_2^2 = K, \\ \kappa_1 = \text{constant} \neq 0, \\ \kappa_2 = \text{constant}, \\ \kappa_2\kappa_3 = 0. \end{cases}$$

Then, we calculate $(\nabla_{\gamma'}^N \nabla_{\gamma'}^N)^k \tau(\gamma)$.

$$\begin{aligned} (\nabla_{\gamma'}^N \nabla_{\gamma'}^N)^k \tau(\gamma) &= (-1)^k \kappa_1 (\kappa_1^2 + \kappa_2^2)^k e_2 \\ &= (-1)^k \kappa_1 K^k e_2. \end{aligned}$$

So, we have

$$\begin{aligned} &(\nabla_{\gamma'}^N \nabla_{\gamma'}^N)^{k-1} \tau(\gamma) + K \{ (\nabla_{\gamma'}^N \nabla_{\gamma'}^N)^{k-2} \tau(\gamma) - \langle \gamma', (\nabla_{\gamma'}^N \nabla_{\gamma'}^N)^{k-2} \tau(\gamma) \rangle \gamma' \} \\ &= (-1)^{k-1} \kappa_1 K^{k-1} e_2 + K (-1)^{k-2} \kappa_1 K^{k-2} e_2 \\ &= 0. \end{aligned}$$

And harmonic is always k -harmonic. So we have Theorem 2.7. □

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